JOURNAL OF COMPUTATIONAL PHYSICS **11**, 240–249 (1973)

Solution of the Equations of Motion of a Charged Particle in a Stationary Magnetic Field

M. METCALF AND M. REGLER

CERN, Geneva, Switzerland

Received June 20, 1969

It is often necessary in high-energy physics experiments to be able to make fast calculations of the paths of charged particles in a magnetic field, for both Monte Carlo and reconstruction programs. If it is possible to represent the field analytically in a Cartesian coordinate system by nonnegative powers of x_1, x_2, x_3, then the differential equation of motion may be solved by comparison of coefficients.

1. METHOD

The equation of motion of a particle in a magnetic field has the form

$$x''_\alpha = f \sum_{\beta=1}^{3} \sum_{\gamma=1}^{3} \epsilon_{\alpha\beta\gamma} x_\beta'(t)\, B_\gamma(x_1(t), x_2(t), x_3(t)),$$

where

f is a constant factor

$'$ denotes d/dt (where t can be the time, the path length, the proper time, etc.)

$$\epsilon_{\alpha\beta\gamma} = \begin{cases} 1, & \text{when } \alpha, \beta, \gamma \text{ is an even permutation of } 1, 2, 3, \\ 0, & \text{when at least two indices are equal}, \\ -1, & \text{when } \alpha, \beta, \gamma \text{ is an odd permutation of } 1, 2, 3. \end{cases}$$

B must be given analytically with nonnegative powers in the Cartesian coordinates $x_\alpha (\alpha = 1, 2, 3)$ in the considered region, and must not depend explicitly on the parameter t [1]:

$$B_\gamma(\mathbf{x}) = \sum_{l=1}^{N_\gamma} C_\gamma(l)\, x_1^{i_\gamma(l)} x_2^{j_\gamma(l)} x_3^{k_\gamma(l)}.$$

We make the following ansatz for the trajectory:

$$x_\alpha(t) = \sum_{h=1}^{\infty} a_\alpha(h)\, t^{h-1}$$

and obtain [2]

$$[x_\alpha(t)]^m = \sum_{q=1}^{\infty} d_{\alpha,m}(q)\, t^{q-1},$$

where

$$d_{\alpha,m}(q) = \begin{cases} 0 & \text{for } \; q \leqslant n_\alpha m \\ [a_\alpha(n_\alpha + 1)]^m & \text{for } \; q = n_\alpha m + 1 \\ \dfrac{1}{(q - n_\alpha m - 1)\, a_\alpha(n_\alpha + 1)} \displaystyle\sum_{p=1}^{q-n_\alpha m-1} [p(m+1) - q + n_\alpha m + 1] & \\ \times\, a_\alpha(p + n_\alpha + 1)\, d_{\alpha,m}(q - p) & \text{for } \; q > n_\alpha m + 1, \end{cases}$$

$a_\alpha(n_\alpha + 1)$ being the first nonzero coefficient in x_α. Further,

$$x_1^{i_\gamma(l)} x_2^{j_\gamma(l)} x_3^{k_\gamma(l)} = \sum_{p=1}^{\infty} K_{\gamma,l}(p)\, t^{p-1},$$

where

$$K_{\gamma,l}(p) = \sum_{q_1+q_2+q_3=p+2} d_{1,i_\gamma(l)}(q_1)\, d_{2,j_\gamma(l)}(q_2)\, d_{3,k_\gamma(l)}(q_3) \quad (q_i \geqslant 1)$$

giving

$$\begin{aligned} B(\mathbf{x}(t))_\gamma &= \sum_{l=1}^{N_\gamma} C_\gamma(l) \sum_{p=1}^{\infty} K_{\gamma,l}(p)\, t^{p-1} \\ &= \sum_{p=1}^{\infty} \sum_{l=1}^{N_\gamma} C_\gamma(l)\, K_{\gamma,l}(p)\, t^{p-1} \\ &= \sum_{p=1}^{\infty} G_\gamma(p)\, t^{p-1}, \end{aligned}$$

where

$$G_\gamma(p) = \sum_{l=1}^{N_\gamma} C_\gamma(l)\, K_{\gamma,l}(p).$$

The a_α may now be calculated recursively from the given initial conditions

$$x_\alpha |_{t=0} = a_\alpha(1), \; x_\alpha' |_{t=0} = a_\alpha(2), \text{ where } \alpha = 1, 2, 3.$$

We obtain by differentiation

$$x_\alpha(t)'' = \sum_{h=1}^{\infty} a_\alpha(h)(h-1)(h-2)\, t^{h-3}$$

$$= \sum_{h=1}^{\infty} h(h+1)\, a_\alpha(h+2)\, t^{h-1}$$

$$x_\beta(t)' = \sum_{h=1}^{\infty} a_\beta(h)(h-1)\, t^{h-2}$$

$$= \sum_{h=1}^{\infty} h a_\beta(h+1)\, t^{h-1}.$$

Therefore, by comparison of coefficients,

$$\sum_{h=1}^{\infty} h(h+1)\, a_\alpha(h+2)\, t^{h-1}$$

$$= f \sum_{\beta=1}^{3} \sum_{\gamma=1}^{3} \epsilon_{\alpha\beta\gamma} \left(\sum_{h=1}^{\infty} h a_\beta(h+1)\, t^{h-1}\right)\left(\sum_{h=1}^{\infty} G_\gamma(h)\, t^{h-1}\right).$$

We write

$$\left(\sum_{h=1}^{\infty} h a_\beta(h+1)\, t^{h-1}\right)\left(\sum_{h=1}^{\infty} G_\gamma(h)\, t^{h-1}\right)$$

$$= \sum_{h=1}^{\infty} \sum_{h_1+h_2=h+1} h_1 a_\beta(h_1+1)\, G_\gamma(h_2)\, t^{h-1} = \sum_{h=1}^{\infty} H_{\beta\gamma}(h)\, t^{h-1} \qquad (h_i \geqslant 1),$$

where

$$H_{\beta\gamma}(h) = \sum_{h_1+h_2=h+1} h_1 a_\beta(h_1+1)\, G_\gamma(h_2),$$

so that

$$\sum_{h=1}^{\infty} h(h+1)\, a_\alpha(h+2)\, t^{h-1} = f \sum_{h=1}^{\infty} \sum_{\beta=1}^{3} \sum_{\gamma=1}^{3} \epsilon_{\alpha\beta\gamma} H_{\beta\gamma}(h)\, t^{h-1}$$

and finally

$$a_\alpha(h+2) = \frac{f}{h(h+1)} \sum_{\beta=1}^{3} \sum_{\gamma=1}^{3} \epsilon_{\alpha\beta\gamma} H_{\beta\gamma}(h) \qquad (h \geqslant 1).$$

2. Considerations on the Convergence

2.1. *Homogeneous Field*

B has the form

$$(B, 0, 0)$$

and the equations of motion read

$$x_1'' = 0, \qquad x_2'' = fx_3'B, \qquad x_3'' = -fx_2'B.$$

For simplification, let

$$u = t/r, \qquad r = 1/fB,$$

where r denotes the radius of the trajectory;

$$b_\alpha(1) = a_\alpha(1), \qquad b_\alpha(2) = ra_\alpha(2).$$

These changes give the simple set of equations

$$\ddot{x}_1 = 0, \qquad \ddot{x}_2 = \dot{x}_3, \qquad \ddot{x}_3 = -\dot{x}_2,$$

where the dot denotes the operation d/du.

Using the ansatz for the trajectory equation,

$$\sum_{h=1}^{\infty} h(h+1)\, b_1(h+2)\, t^{h-1} = 0,$$

$$\sum_{h=1}^{\infty} [h(h+1)\, b_2(h+2) - hb_3(h+1)]\, t^{h-1} = 0,$$

$$\sum_{h=1}^{\infty} [h(h+1)\, b_3(h+2) + hb_2(h+1)]\, t^{h-1} = 0,$$

we find by comparison of coefficients

for $h = 1$

$$b_1(3) = 0,$$
$$b_2(3) = \tfrac{1}{2}b_3(2),$$
$$b_3(3) = -\tfrac{1}{2}b_2(2);$$

for $h = 2$

$$b_1(4) = 0,$$
$$b_2(4) = \tfrac{1}{3}b_3(3) = -\tfrac{1}{6}b_2(2),$$
$$b_3(4) = -\tfrac{1}{3}b_2(3) = \tfrac{1}{6}b_3(2), \text{ etc.}$$

Hence

$$x_1(u) = b_1(1) + b_1(2)\,u,$$

$$x_2(u) = b_2(1) + b_2(2)\,u + \tfrac{1}{2}b_3(2)\,u^2 - \tfrac{1}{6}b_2(2)\,u^3 + \cdots$$

$$= b_2(1) + b_2(2) \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k-1)!}\,u^{2k-1} - b_3(2) \sum_{k=2}^{\infty} \frac{(-1)^{k-1}}{(2k-2)!}\,u^{2k-2},$$

$$x_3(u) = b_3(1) + b_3(2)\,u - \tfrac{1}{2}b_2(2)\,u^2 + \tfrac{1}{6}b_3(2)\,u^3 + \cdots$$

$$= b_3(1) + b_3(2) \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k-1)!}\,u^{2k-1} + b_2(2) \sum_{k=2}^{\infty} \frac{(-1)^{k-1}}{(2k-2)!}\,u^{2k-2}.$$

The sums contained in these terms are the $\sin(u)$ or $[\cos(u) - 1]$ series, and therefore converge for any u and rapidly for $u < 1$ ($u = 1$ corresponds to a deflection by the field of 1 radian).

As an example, the trajectory of a 1.1 GeV/c particle in a 10 kG field (radius therefore $\sim$ 300 cm) was calculated. Taking a total path length of 400 cm and initially a single step, the integration was performed a number of times, each time doubling the number of individual steps. In Fig. 1, Δ is the absolute difference between the approximate and exact end points as a function of the upper limit (highest power of u) of the series. After each step $b_\alpha(2)$ was renormalized.

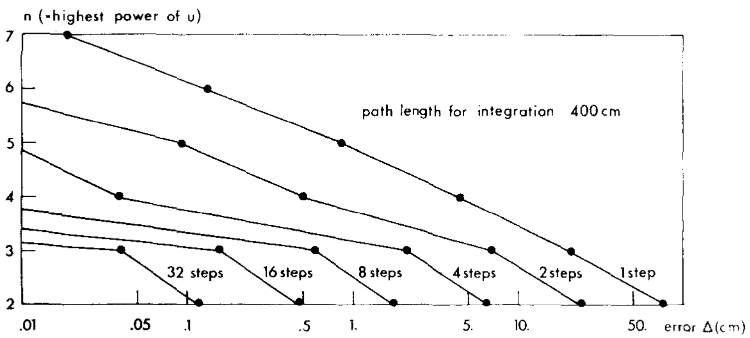

FIGURE 1

For any given accuracy a minimum number of operations is performed in the integrations by making just a single step.

2.2. *Inhomogeneous Field*

The deviation of a calculated trajectory from its true path depends not only on the accuracy of the tracking method, but also on the precision of the field representation. The field model—a finite power series with nonnegative powers in x_1, x_2, x_3—should be kept as simple as possible, subject only to the required precision, since the number of computational operations necessary increases rapidly with the number of field coefficients.

The choice of the highest power n of the trajectory parameter in the series used for the trajectory representation must be made empirically, although, in general, the highest power required will exceed that of the field representation by at least two; if terms of power p in x_α were necessary to achieve a given accuracy in the field representation, this power will also be necessary in order to describe well the field along the trajectory:

$$\mathbf{B}(t) = \sum_{i=1}^{p+1} \frac{1}{(i-1)!} \mathbf{B}^{(i-1)}(0)\, t^{i-1},$$

where

$$\mathbf{B}^{(i-1)}(0) = \left. \frac{d^{(i-1)}\mathbf{B}}{dt^{i-1}} \right|_{t=0}.$$

The derivative $\mathbf{x}^{(n)}$ is the first which contains the $(n-2)$-th power of the field, i.e.,

$$\mathbf{x}''(t) = f\mathbf{x}'(t) \wedge \mathbf{B}(\mathbf{x}(t)),$$

where $\wedge$ defines a vector product, and after differentiating p times with respect to t, and $n = p + 2$,

$$\mathbf{x}^{(n)} = f \sum_{j=1}^{n-1} \binom{n-1}{j} \mathbf{x}^{(n-j)} \wedge \mathbf{B}^{(j-1)}.$$

Should an accuracy thereby be obtained which is greater than that which corresponds to the field representation, then one should not simply cut the series earlier, but simplify the field representation to correspond to the accuracy required in just that part of the field volume through which the particles pass.

2.3. *Two-dimensional field*

If the field is dependent on only two of the three coordinates x_α, and if its representation fulfils Maxwell's equations (which is in general not a necessary condition for the application of the above method), then it is possible to prepare a simple test program which at the same time illustrates the convergence process. If, for example, the field model is independent of x_2, the second equation of the set has the form

$$x_2'' = f\{x_3' B_1(x_1, x_3) - x_1' B_3(x_1, x_3)\},$$

$$(\partial B_1/\partial x_1) + (\partial B_3/\partial x_3) = 0, \qquad (\partial B_1/\partial x_3) - (\partial B_3/\partial x_1) = 0.$$

Integration of this equation gives the variation in x_2':

$$\Delta x_2' = f \int_0^t (x_3' B_1(x_1, x_3) - x_1' B_3(x_1, x_3))\, dt$$

$$= f \int_{x_1(0), x_3(0)}^{x_1(t), x_3(t)} (B_1\, dx_3 - B_3\, dx_1) = \phi(x_1(t), x_3(t), x_1(0), x_3(0)).$$

If one now substitues for $x_1(t)$, $x_3(t)$, and $x_2'(t)$ the ansatz from Section I, the two sides of the equation should converge.

As an example, using the notation in Section I,

$$\begin{array}{llll} C_1(1) = -20, & i_1(1) = 1, & j_1(1) = 0, & k_1(1) = 1, \\ C_3(1) = 10000, & i_3(1) = 0, & j_3(1) = 0, & k_3(1) = 0, \\ C_3(2) = -10, & i_3(2) = 2, & j_3(2) = 0, & k_3(2) = 0, \\ C_3(3) = 10, & i_3(3) = 0, & j_3(3) = 0, & k_3(3) = 2, \end{array}$$

$$B_1 = -20 x_1 x_3,$$
$$B_3 = 10000 - 10 x_1^2 + 10 x_3^2,$$

$$\Delta x_2' = -10000 f\{x_1(t) - x_1^3(t)/3000 + x_1(t)\, x_3^2(t)/1000 - x_1(0) + x_1^3(0)/3000 - x_1(0)\, x_3^2(t)/1000 + (x_1(0)\, x_3^2(t) - x_1(0)\, x_3^2(0))/1000\}.$$

TABLE I

Convergence of the "deflexion"

n	α[a]	β[b]
1	−0.0000	−0.1901
2	−0.1036	−0.1901
3	−0.1462	−0.1897
4	−0.1898	−0.1896
5	−0.1892	−0.1891
6	−0.1893	−0.1889
7	−0.1896	−0.1889
8	−0.1891	−0.1889
9	−0.1890	−0.1889
10	−0.1889	−0.1889
∞	−0.1889	−0.1889

a) $\alpha = \left(\sum_{h=1}^{n+1} (h-1)\, a_2(h) t^{h-2}\right) - a_2(2).$

b) $\beta = \phi\left(\sum_{h=1}^{n+1} a_1(h) t^{h-1}, \sum_{h=1}^{n+1} a_3(h) t^{h-1}, a_1(1), a_3(1)\right).$

For the initial conditions,

$$a_1(1) = x_1(0) = -10.0, \qquad a_2(1) = x_2(0) = 0, \qquad a_3(1) = x_3(0) = 0,$$
$$a_1(2) = x_1'(0) = 0.8, \qquad a_2(2) = x_2'(0) = 0, \qquad a_3(2) = x_3'(0) = 0.6,$$

and with

$$t = 50.0, \qquad f = 2.998 \times 10^{-7},$$

we find the values of Table I.

The field representation goes up to the power of two, whereas the fourth-order representation of the trajectory gives the "deflexion" to a satisfactory accuracy.

3. Application

3.1. *A Specific Magnet*

The Split Field Magnet (SFM) [3] at CERN was chosen as an example of a magnet having a very inhomogeneous field, with a maximum value of 10 kG and a volume of 350 × 1240 × 110 cm^3. The field has been analysed, for the sake of simplicity of use, by dividing one quadrant into 1152 equal boxes and determining the coefficients of polynomial series satisfying Maxwell's equations for the field components at a point [4]. No test on the significance of the coefficients was made. The series were of power one, two, or three, the program seeking the minimum order fit in each box such that the maximum residual at a measured point did not exceed 30 G. This represents, for a 1.1 GeV/c particle at the center of the detector system, an error in the tracking due to an error in the determination of the field which is of the same order as the error due to the resolution of wire spark chambers placed in the field.

Two FORTRAN tracking programs have been written and tested, and compared with a third standard program using the Runge–Kutta method [5]. The first of the two programs uses a subroutine EQUMOT [6], which contains the equations of Section I written implicitly, together with a subroutine to prepare the field coefficients in the required form. The processing time of this program rises rapidly with the order of the tracking, but it allows one to obtain any desired accuracy.

In the second program, use was made of a subroutine TRACE in which the equations are written out explicitly for a field of maximum power three. To facilitate the writing of this program, which is rather intricate owing to the large number of terms involved and the need to factorize them as well as possible, the field representation was transformed into a system with its origin at the starting point of the integration path using a separate subroutine FIELD, written for the purpose. TRACE, of course, gives identical results to those of EQUMOT, whilst

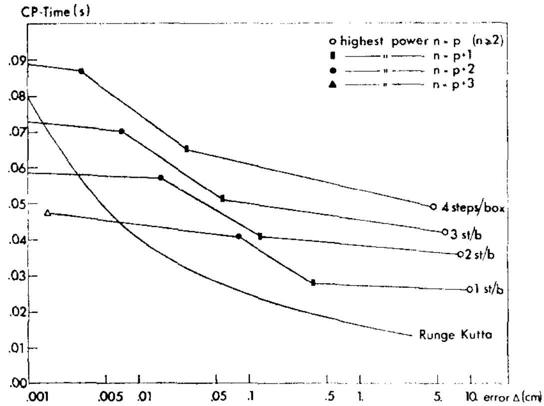

FIGURE 2

being typically a factor 20 times faster. Figure 2 is a graph of central processor time (CDC 6600) versus the accuracy obtained by TRACE for a 1.1 GeV/c particle followed from the origin to the center of the group of detectors. The linked points give curves for different choices of path length; the different symbols indicate the tracking order used in each box traversed by the particle, this being a function of the highest power of the field representation in the box. For comparison, the Runge–Kutta curve is included.

For low accuracy the latter method is faster, this being due to the limitation imposed on the path length in TRACE by the fact that the box size was kept small enough not to require a fit to a power greater than three, beyond which programming becomes impractical, whilst it is evident from Fig. 2 that even in such an inhomogeneous field a larger path length should be chosen. This demonstrates a competitive situation which arises when a compromise has been reached between storage space and a simple field model.

3.2. *A Specific Problem Requiring Tracking to a Set of End Points for Fixed Initial Conditions*

The problem arises in the central region of the SFM of performing very fast tracking towards the center of the magnet. A TRACE-like program has been written which performs the tracking from the edge of this central region to its center in a single step, which takes 1.74 msec (CDC 6600). By reusing the initial conditions and adjusting the path length one can approach the desired end point iteratively, these subsequent calls each taking only 0.126 msec more.

4. Conclusions

The method described is well adapted to an homogeneous field.

In the case of an inhomogeneous field one must first determine how much fast storage is available; for a rather fine grid of field measurements the Runge–Kutta method is faster, whereas if it is necessary to comply with more limiting storage considerations by using an analytical model of the field, the time required to calculate field values becomes a dominant part of the Runge–Kutta time. In this case the method described in this paper is to be preferred.

If, as above, a compromise has been reached between storage space and a simple model, the two methods are competitive.

The above method is especially well adapted to problems requiring the calculation of a large number of end points from a single set of initial conditions. In this case the coefficients $a_\alpha(h)$ from the trajectory equations need to be calculated only once.

In addition, the method allows the calculation of the intersection points of a trajectory with a given surface $G(x_1, x_2, x_3) = 0$ (particle detectors, for example). Having constant initial conditions, one can use an iterative procedure to approach the surface, thus obviating the necessity to solve high-order equations to define the path length to the intersection point.

Acknowledgments

We would like to thank Dr. K. Winter for his continuing support of our work, and Drs. H. Wind and T. Pomentale for helpful discussions. Miss F. Ranjard and Dr. H. Grote have afforded us much practical assistance.

We would particularly like to thank Mr. K. S. Kölbig for his valuable advice concerning the preparation of this paper.

References

1. H. Wind, *J. Computational Phys.* **2** (1968), 274.
2. I. S. Gradshteyn and I. M. Ryzhik, "Table of Integrals, Series, and Products," Academic Press, New York, 1965.
3. A. Minten, "The Split Field Magnet Facility," CERN-SFMD-Note-4, 28 April 1971.
4. M. Metcalf, "Analysis of the SFM Field," CERN-OM Development Note AP-10, 8 November 1971.
5. M. Abramowitz, "Handbook of Mathematical Functions," Dover, New York, 1965.
6. M. Regler, EQUMOT — "Equations of Motion of a Particle in a Static Magnetic Field," CERN Computer Program Library W306, 1968.